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Riemann problem for the zero-pressure flow in gas dynamics*

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Abstract The Riemann problem for zero-pressure flow in gas dynamics in one dimension and two dimensions is investigated. Through studying the generalized Rankine-Hugoniot conditions of delta-shock waves, the one-dimensional Riemann solution is proposed which exhibits four different structures when the initial density involves Dirac measure. For the two-dimensional case, the Riemann solution with two pieces of initial constant states separated at a smooth curve is obtained.

Keywords: zero-pressure flow in gas dynamics, Riemann problem, delta-shocks, generalized Rankine-Hugoniot condition, entropy condition.

Consider a zero-pressure flow in gas dynamics

$$\begin{cases} \rho_t + \operatorname{div}(\rho \boldsymbol{u}) = 0, \\ (\rho u_j)_t + \sum_{k=1}^{d} \dfrac{\partial}{\partial x_k}(\rho u_j u_k) = 0, \end{cases} \qquad j = 1, \cdots, d, \tag{1}$$

where $\rho(x,t) \geqslant 0$ and $\boldsymbol{u}(x,t) = (u_1, \cdots, u_d)(x \in R^d,\ t \in [0, \infty))$ are the density and the mean velocity respectively, $\dfrac{\partial}{\partial x_k}$ represents the partial derivatives with respect to the space component $x_k(k = 1, \cdots, d)$. This model can be viewed as a direct result of imposing the pressure $p = 0$ on the Euler equations,

$$\begin{cases} \rho_t + \operatorname{div}(\rho \boldsymbol{u}) = 0, \\ (\rho u_j)_t + \sum_{k=1}^{d} \dfrac{\partial}{\partial x_k}(\rho u_j u_k) + \dfrac{\partial p}{\partial x_j} = 0, \end{cases} \qquad j = 1, \cdots, d. \tag{2}$$

Besides, model (1) is referred to as the adhesion particle dynamics system to explain the formation of the large-scale structures in the universe[1]. In this paper, d is 1 or 2.

The distinct feature is that Eqs. (1) have repeated eigenvalues and incomplete associated eigenvectors. The C^1 solution of (1) can be given by the explicit formula

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$$\begin{cases} \boldsymbol{u}(x,t) = \boldsymbol{u}_0(\psi(x,t)), \\ \rho(x,t) = \dfrac{\rho_0(\psi(x,t))}{\left|\sigma_{ik} + t\dfrac{\partial u_j}{\partial x_k}\right|}, \end{cases} \tag{3}$$

where $(\rho_o, u_0)(x_0) \in C^1(R^d)$ are the initial density and veloctiy, $\psi(x,t)$ is the inverse of flow map $I + tu_0 : x = x_0 + tu_0(x_0)$ and $\sigma_{ik} = \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases}$ the Kronecker symbol.

Equation (3) shows that C^1 solutions of (1) may not exist after a finite time. More precisely, the density itself and the gradient of velocity become a singular measure if the flow map is compressive. So we have to seek discontinuous solutions. For bounded BV solutions, the normal directional components of the velocity on both sides of a discontinuity must be identical, otherwise there is no bounded solution. Noticing that the solution must develop Dirac measure in the density for some cases, we introduce delta-shocks in the solution of (1). For the exact definition of the delta-shock wave, the reader may refer to Reference [2].

The study of delta-shocks began in 1990 when Tan et al. found that no classical weak solution existed for some values of initial data in the investigation of the Reimann problem for a 2×2 simplified model of the Euler equations[3],

$$\begin{cases} u_t + (u^2)_x + (uv)_y = 0, \\ v_t + (uv)_x + (v^2)_y = 0, \end{cases} \tag{4}$$

and that delta-shock waves were necessarily used as parts in their solutions. Since then some efforts have been made to search for the physical background of the delta-shock and to perfect its mathematical theory. Sheng et al.[4] set forth from the splitting method of the numerical scheme of the Euler equation, and studied the Riemann problem for (1) in one dimension and in two dimensions with the vanishing viscosity method when the initial density is just a function of total bounded variation. Later it was found that (1) is just the consequence of neglecting the pressure effect on the Euler equation1) or an adhesion particle dynamics model in astrophysics[1]. The delta-shock can be explained as the concentration of particles after collision. Besides, Bouchut also discussed this model in one dimension from the point of view of cold plasma, low pressure and evanescent viscosity[5].

Since the delta-shocks appear in the solution, it is natural to consider that when the initial data contains Dirac measures, the viscosity vanishing method is too complicated to solve this problem. Thus we gave generalized Rankine-Hugoniot conditions to define one-dimensional and two-dimensional delta-shocks[2]. Although the generalized Rankine-Hugoniot conditions are rather complicated in form, we can discuss them explicitly.

1) Li, J. Note on the compressible Euler equations with zero temperature. Appl. Math. Let., in press, 2000.

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In this paper we study the Riemann problem for Eqs.(1) in one dimension and two dimensions respectively. The crucial arguments are how to solve the generalized Rankine-Hugoniot conditions. The above discussion shows that the Riemann solutions can be used as a building block to establish the existence of the general Cauchy problem.

1 Generalized Rankine-Hugoniot conditions of delta-shocks

In this section we will give a precise definition of solutions of (1) in the sense of distributions. The equivalent solution can be defined in the sense of measure. Then we will offer the generalized Rankine-Hugoniot conditions of delta-shocks and the so-called entropy conditions to ensure the uniqueness.

Definition 1. $(\rho, \boldsymbol{u})$ is a solution of (1) in the sense of distributions if

$$\begin{cases} \displaystyle\iint_{[0,\infty)\times R^d} \rho\varphi_t + (\rho\boldsymbol{u})\cdot\nabla\varphi \,\mathrm{d}x\mathrm{d}t = 0, \\ \displaystyle\iint_{[0,\infty)\times R^d} \rho u_j\varphi_j + (\rho u_j\boldsymbol{u})\cdot\nabla\varphi \,\mathrm{d}x\mathrm{d}t = 0, \end{cases} \qquad j = 1, d; d = 1 \text{ or } 2 \tag{5}$$

for every $\varphi(t,x)\in C_C^\infty([0,\infty)\times R^d)$, where ∇ is the gradient operator.

Definition 2. The weighted Dirac delta function $w\delta$ is a distribution in $D'([c,d]\times R^d)$.

(i) $w(t)\delta\in D'([c,d]\times R)$ supported on a smooth curve $L: x = x(t)(c\leqslant t\leqslant d)$ is defined by

$$\langle w(t)\delta_L, \varphi\rangle = \int_c^d w(t)\varphi(t,x(t))\mathrm{d}t, \qquad \varphi(t,x)\in C_c^\infty([0,\infty]\times R). \tag{6}$$

(ii) $w(t,s)\delta_S\in D'([c,d]\times R^2)$ supported on a smooth surface of

$$S:\begin{cases} x = x(t,s), \\ y = y(t,s), \end{cases} \qquad (a\leqslant t\leqslant b,\ c\leqslant s\leqslant d)$$

is define by

$$\langle w\delta_S, \varphi\rangle = \int_a^b\int_c^d w(t,s)\varphi(t,x(t,s),y(t,s))\mathrm{d}t\mathrm{d}s,$$
$$\varphi(t,x,y)\in C_c^\infty([0,\infty)\times R^2). \tag{7}$$

We consider the one-dimensional case of (1) at the outset. For a bounded discontinuity $x = x(t)$ in the sense that the solution is BV, the Rankine-Hugoniot condition is

$$\sigma = \frac{\mathrm{d}x}{\mathrm{d}t} = l_- u(t) = l_+ u(t). \tag{8}$$

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Throughout this paper $l_- u$ and $l_+ u$ are the limit values of u on the left-hand side and right-hand side of $x = x(t)$ respectively.

Identity (8) shows that the bounded discontinuity is just a slip line denoted by J, and two states, $(\rho_l, u_l)(x, t)$ and $(\rho_r, u_r)(x, t)$, can be connected by J if and only if $l_- u(t) = l_+ u(t)$. When the initial data of Eqs. (1) are decreasing, the characteristic lines overlap and bounded solutions do not exist. So we introduce delta-shock solutions which are measure solutions in the form of

$$(\rho, u)(x, t) = \begin{cases} (\rho_l, u_l)(x, t), & x < x(t), \\ (w(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\ (\rho_r, u_r)(x, t), & x > x(t), \end{cases} \tag{9}$$

where $x = x(t)$ is smooth enough. This solution consists of two parts, an ordinary part and a singular part. It should be understood in the sense of distributions. The reason why they are written in such a form is that we can see where the ordinary and singular parts are without introducing more notations. The same notations will often be used in the following such as Eqs. (14) and (18) below.

It is easy to check from Eqs. (5) that $x(t)$, $u_\delta(t)$ and $w(t)$ are defined via the generalized Rankine-Hugoniot condition

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u_\delta, \frac{\mathrm{d}w}{\mathrm{d}t} = [\rho] u_\delta - [\rho u], \frac{\mathrm{d}(wu_\delta)}{\mathrm{d}t}[\rho u] u_\delta - [\rho u^2], \tag{10}$$

where $[p] = p_r - p_l$ is the jump quantity of p across the delta-shock $x = x(t)$ for which we supplement the so-called entropy condition

$$l_- u(t) > u_\delta(t) > l_+ u(t). \tag{11}$$

Here we should point out that the entropy inequality

$$\frac{\partial}{\partial t}\rho S(u) + \frac{\partial}{\partial x}\rho u S(u) \leqslant 0 \tag{12}$$

for every convex function $S: R \to R$ cannot ensure the uniqueness[5].

For the two-dimensional case of (1), if we consider bounded weak solutions as similar to those of the one-dimensional case, then we obtain the Rankine-Hugoniot condition of

$$n_t = -u_l n_x - v_l n_y = -u_r n_x - v_r n_y \tag{13}$$

for a bounded discontinuity, where (n_t, n_x, n_y) is the normal direction of the discontinuity. Identity (13) shows that the components of velocity in the normal direction of a discontinuity on both sides of the discontinuity must be consistent. Therefore it is necessary to consider delta-shock solutions as similar to those in the one-dimensional case.

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Let a delta-shock solution be

$$(\rho,u,v)(x,y,t)=\begin{cases}(\rho_l,u_l,v_l)(x,y,t),\\(w(t,s)\delta(x-x(t,s),y-y(t,s)),u_\delta(t,s),v_\delta(t,s)),\\(\rho_r,u_r,v_r)(x,y,t),\end{cases}\tag{14}$$

where

$$S:\begin{cases}x=x(t,s),\\y=y(t,s)\end{cases}\tag{15}$$

is a discontinuity surface with a suitable parameter s. Then the generalized Rankine-Hugoniot condition is a system of the first-order partial differential equations

$$\begin{cases}\dfrac{\partial x}{\partial t}=u_\delta(t,s),\\[2ex]\dfrac{\partial y}{\partial t}=v_\delta(t,s),\\[2ex]\dfrac{\partial w}{\partial t}=([\rho],[\rho u],[\rho v])\cdot(n_t,n_x,n_y),\end{cases}\tag{16a}$$

$$\begin{cases}\dfrac{\partial(wu_\delta)}{\partial t}=([\rho u],[\rho u^2],[\rho uv])\cdot(n_t,n_x,n_y),\\[2ex]\dfrac{\partial(wv_\delta)}{\partial t}=([\rho v],[\rho uv],[\rho v^2])\cdot(n_t,n_x,n_y),\end{cases}\tag{16b}$$

where u and v are the respective components of velocity in x and y directions, $(n_t,\ n_x,\ n_y)=\left(u_\delta\dfrac{\partial y}{\partial s}-v_\delta\dfrac{\partial x}{\partial s},-\dfrac{\partial y}{\partial s},\dfrac{\partial x}{\partial s}\right)$ is the normal direction of S oriented from (l) to (r) ((l) is the abbreviation of the side where the state is (ρ_l,u_l,v_l)), $[\rho]=\rho_l-\rho_r$. For the definiteness, we always denote by (r) the side of $(x-x_0,\ y-y_0)\cdot(n_x,\ n_y)>0$ for all $t>0$, where (x_0,y_0,t) is a point of S.

For the two-dimensional delta-shock S, the geometrical entropy condition is that the characteristic lines on both sides of S are incoming. In other words,

$$(u_r,v_r)\cdot(n_x,n_y)<(u_\delta,v_\delta)\cdot(n_x,n_y)<(u_l,v_l)\cdot(n_x,n_y).\tag{17}$$

As we have pointed out above, system (16) has exactly a quintuple eigenvalue $\lambda=0$ independent of all variables. In this sense, it is somewhat similar to a degenerately linear system of hyperbolic partial differential equations.

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2 Riemann solution in one dimension

Now we start to discuss the Riemann problem for (1) in one dimension with the standard characteristic method using the generalized Rankine-Hugoniot condition (10). Due to obvious reasons, we consider the Riemann initial data involving singular measures

$$(\rho, u)(x, 0) = \begin{cases} (\rho_l, u_l), & x < 0, \\ (m_0\delta, u_0), & x = 0, \\ (\rho_r, u_r), & x > 0, \end{cases} \tag{18}$$

where ρ_l, m_0 and ρ_r are not all zero. Otherwise the solution is trivial.

At first, we solve system (10) with the initial condition

$$x(0) = 0, w(0) = m_0 \quad \text{and} \quad u_\delta(0) = u_0 \tag{19}$$

satisfying $u_l > u_0 > u_r$. Obviously, we have

$$\begin{aligned} w - [\rho]x &= m_0 - [\rho u]t, \\ wu_\delta - [\rho u]x &= m_0 u_0 - [\rho u^2]t. \end{aligned} \tag{20}$$

It follows that

$$[\rho]xu_\delta - [\rho u]x = m_0 u_0 - [\rho u^2]t - (m_0 - [\rho u]t)u_\delta \tag{21a}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{[\rho]}{2}x^2 + (m_0 - [\rho u]t)x\right) = m_0 u_0 - [\rho u^2]t. \tag{21b}$$

Solving this equation gives

$$x(t) = \begin{cases} \dfrac{m_0 u_0 t - t^2[\rho u^2]/2}{m_0 - [\rho u]t}, & \rho_l = \rho_r, \\[2ex] \dfrac{-(m_0 - [\rho u]t) + w(t)}{[\rho]}, & \rho_l \neq \rho_r, \end{cases} \tag{22}$$

and

$$w(t) = (m_0^2 + 2m_0([\rho]u_0 - [\rho u])t + \rho_l\rho_r(u_l - u_r)^2 t^2)^{\frac{1}{2}}. \tag{23}$$

Furthermore, from (20) we get

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$$u_\delta(t) = \frac{1}{w(t)}([\rho u]x + m_0 u_0 - [\rho u^2)]t). \tag{24}$$

Especially, if $m_0 = 0$, then

$$(x(t), w(t), u_\delta(t)) = \left(\frac{\sqrt{\rho_l}u_l + \sqrt{\rho_r}u_r}{\sqrt{\rho_l} + \sqrt{\rho_r}} t, \sqrt{\rho_l \rho_r}(u_l - u_r)t, \frac{\sqrt{\rho_l}u_l + \sqrt{\rho_r}u_r}{\sqrt{\rho_l} + \sqrt{\rho_r}} \right). \tag{25}$$

Lemma 1. The solutions of (10) and (19) possess the following properties:

(i) $u_r < u_\delta < u_l$ (26)

(ii) If $\rho_l + \rho_r > 0$, then

$$\lim_{t \to \infty} u_\delta(t) = \frac{\sqrt{\rho_l}u_l + \sqrt{\rho_r}u_r}{\sqrt{\rho_l} + \sqrt{\rho_r}}. \tag{27}$$

If $\rho_l = \rho_r = 0$, then $u_\delta \equiv u_0$.

(iii) $x'(t)$ is a monotone function of t.

The proof of this Lemma is a simple matter, so the details are omitted.

Now we start to construct the Riemann solutions of (1) and (18). For simplicity, we hereinafter denote a particular particle by its Lagrange coordinate x_0, and the vacuum state by Vac. According to the relation of u_l, u_0 and u_r, we solve the Riemann problem case by case.

Case 1. $u_r < u_0 < u_l$. This is a typical case, a delta-shock emits from the origin. Solving (10) and (19), we obtain a fusion solution

$$(\rho, u)(x, t) = \begin{cases} (\rho_l, u_l), & x < x(t), \\ (w(t), \delta, u_\delta(t)), & x = x(t), \\ (\rho_r, u_r), & x > x(t), \end{cases} \tag{28}$$

where $x(t)$, $w(t)$ and $u_\delta(t)$ are defined in Equations (22) ~ (24).

Case 2. $u_l \geqslant u_r \geqslant u_0$ (for the case where $u_0 \geqslant u_l \geqslant u_r$, the structure of solution is similar).

It is easy to see that the particles at $x_0 < 0$ collide with the particle at the origin at first, and the trajectory, mass and velocity of the fused particles are respectively

$$x = x_l(t) = \frac{m_0 + \rho_p u_l t - (m_0^2 + m_0 \rho_l (u_l - u_0)t)^{\frac{1}{2}}}{\rho_l},$$

$$w = w_l(t) =)m_0^2 + 2m_0 \rho_l (u_l - u_0)t)^{\frac{1}{2}},$$

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$$u_{\delta_1}(t) = u_l - \frac{m_0(u_l - u_0)}{w(t)}. \tag{29}$$

At the time t^* while $x_1(t^*) = u_r t^*$, i.e.

$$t = t^* = \frac{2m_0(u_r - u_0)}{\rho_l(u_l - u_r)^2}, \tag{30}$$

the particles at $x_0 \leqslant 0$ start to collide with those with $x_0 > 0$. At this moment,

$$x = x^* = x_l(t^*), \quad w = w^* = w_l(t^*), \quad u_\delta = u_\delta^* = u_{\delta_1}(t^*). \tag{31}$$

After that, $x(t)$, $w(t)$ and $u_\delta(t)$ are defined by the generalized Rankine-Hugoniot condition (10) and the initial condition (31). The details are omitted. The solution can be expressed as

$$(\rho, u)(x,t) = \begin{cases} (\rho_l, u_l), & x < x_1(t), \\ (w_1(t)\delta, u_{\delta_1}(t)), & x = x_1(t), \\ \text{Vac}, & x_1 < x < u_r t, \\ (\rho_r, u_r), & x > u_r t, \end{cases} \tag{32}$$

when $t < t^*$, and

$$(\rho, u)(x,t) = \begin{cases} (\rho_l, u_l), & x < x(t), \\ (w(t)\delta, u_\delta(t)), & x = x(t), \\ (\rho_r, u_r), & x > x(t), \end{cases} \tag{33}$$

when $t \geqslant t^*$.

Case 3. $u_0 < u_l < u_r$ (when $u_l < u_r < u_0$, the structure of solution is similar).

At the beginning, the particles at $x_0 < 0$ collide with the particle at $x_0 = 0$. The trajectory $x(t)$, mass $w(t)$, and velocity $u_\delta(t)$ of the fused particles can be expressed as in (29) ~ (31), respectively,

$$x(t) = x_1(t), \quad w(t) = w_1(t) \quad \text{and} \quad u_\delta(t) = u_{\delta_1}(t). \tag{34}$$

Since $u_l < u_r$ and $u_0 < u_{\delta_1}(t) < u_l$, $u_{\delta_1}(t)$ are always less than u_r, showing that the particles with $x_0 \leqslant 0$ never collide with those $x_0 > 0$. The solution for this case can be expressed as

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$$(\rho,u)(x,t)=\begin{cases}(\rho_l,u_l), & x<x(t),\\ (w(t)\delta(x-x)(t)),u_\delta(t)), & x=x(t),\\ \text{Vac}, & x(t)<x<u_rt,\\ (\rho_r,u_r), & x\geqslant u_rt.\end{cases} \tag{35}$$

Case 4. $u_l\leqslant u_0\leqslant u_r$. For this case, the solution is simple and can be written as

$$(\rho,u)(x,t)=\begin{cases}(\rho_l,u_l), & x\leqslant u_l,t,\\ \text{Vac}, & u_lt<x<u_0t,\\ (m_0\delta(x-u_0(t)),u_0), & x=u_0t,\\ \text{Vac}, & u_0t<x<u_rt,\\ (\rho_r,u_r), & x\geqslant u_rt.\end{cases} \tag{36}$$

This solution is called a collisionless solution.

Remark. When $m_0=0$, the solution exhibits a simple structure which is consistent with the results in Ref. [4]. This implies that the solution constructed here is stable under some viscosity perturbations.

3 Two-dimensional Riemann problem with two pieces of initial data

Now we consider the 2-D Riemann problem for (1) with two pieces of Riemann initial data:

$$(\rho,u,v)(x,y,0)=\begin{cases}(\rho_1,u_1,v_1), & y>f(x),\\ (\rho_2,u_2,v_2), & y<f(x).\end{cases} \tag{37}$$

The crucial argument is to parameterize the discontinuity surface in the form of (15).

For simplicity, we assume that $\Gamma: y=f(x)$ is a smooth curve separating (x,y)-plane into two infinite parts and does not intersect itself. Orient the normal direction of Γ in (x,y)-plane from state 1 to state 2 and denote it by $\boldsymbol{n}=\dfrac{1}{\sqrt{1+f'(x)^2}}(-f'(x),1)$ where (i) denotes the domain in which the initial state is (ρ_i,u_i,v_i) $(i=1,2)$. Then we have the following lemma.

Lemma 2. If $[v]-[u]f'(x)>0$, $\qquad (38)$

then the characteristic lines from state 1 will intersect those from state 2, where $[v]=v_2-v_1$, etc. Otherwise they will not intersect.

Proof. Inequality (38) is equivalent to

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$$u_2f'(x) - v_2 < u_1f'(x) - v_1. \tag{39}$$

The characteristic directions of the left state (ρ_1, u_1, v_1) and the right state (ρ_2, u_2, v_2) are in (x, y, t)-space,

$$c_1 = (u_1, v_1, 1) \quad \text{and} \quad c_2 = (u_2, v_2, 2), \tag{40}$$

and the tangential direction at any point of Γ in (x, y)-plane is

$$\boldsymbol{T} = (1, f'(x), 0). \tag{41}$$

Via computation we obtain

$$\begin{aligned} \boldsymbol{n}_1 &= c_1 \times \boldsymbol{T} = (-f'(x), 1, u_1f'(x) - v_1), \\ \boldsymbol{n}_2 &= c_2 \times \boldsymbol{T} = (-f'(x), 1, u_2f'(x) - v_2). \end{aligned} \tag{42}$$

So the planes passing any point $(s, f(s))$ of Γ and with the normal direction $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are respectively

$$\begin{aligned} \Omega_1: -(x - s)f'(s) + (y - f(s)) + (u_1f'(s) - v_1)t &= 0, \\ \Omega_2: -(x - s)f'(s) + (y - f(s)) + (u_2f'(s) - v_2)t &= 0. \end{aligned} \tag{43}$$

Inequality (39) implies that

$$-(x_1 - s)f'(x) + (y_1 - f(s)) < -(x_2 - s)f'(s) + (y_2 - f(s)), \tag{44}$$

in which (x_1, y_1, t) and (x_2, y_2, t) are points of Ω_1 and Ω_2 respectively. Thus we have

$$(x_2 - x_1, y_2 - y_1) \cdot (f'(s), -1) < 0, \tag{45}$$

which shows that Lemma 2 holds.

Lemma 2 means that a two-dimensional delta-shock wave must emit from the initial discontinuity $\Gamma: y = f(x)$ when (38) is satisfied. In what follows, we will discuss the Riemann problem in two cases.

Case 5. $[v] - [u]f'(x) > 0$.

In order to use the generalized Rankine-Hogoniot condition (16), we parametrize Γ as

$$\Gamma: \begin{cases} x = s, \\ y = f(s), \end{cases} \tag{46}$$

and we just consider the case of $w = 0$ initially and $[\rho] \neq 0$. As the case in which $[\rho] = 0$ is rather simple, we omit it. First, we notice that

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$$[\rho u][\rho v]-[\rho][\rho uv]=\rho_1\rho_2[u][v],$$
$$[\rho][\rho u^2]-[\rho u]^2=-\rho_1\rho_2[u]^2. \tag{47}$$

By the generalized Rankine-Hogoniot condition (16), we have

$$\begin{aligned}[\rho u]\frac{\partial w}{\partial t}-[\rho]\frac{\partial(wu_\delta)}{\partial t}&=([\rho][\rho u^2]-[\rho u]^2)\frac{\partial y}{\partial s}+([\rho v][\rho u]-[\rho][\rho uv])\frac{\partial x}{\partial s}\\&=\rho_1\rho_2[u]\left(-[u]\frac{\partial y}{\partial s}+[v]\frac{\partial x}{\partial s}\right)\end{aligned} \tag{48}$$

and

$$[\rho v]\frac{\partial w}{\partial t}-[\rho]\frac{\partial(wu_\delta)}{\partial t}=-\rho_1\rho_2[v]\left(-[u]\frac{\partial y}{\partial s}+[v]\frac{\partial x}{\partial s}\right). \tag{49}$$

A comparison of (48) with (49) leads to the identity

$$[v]\left([\rho u]\frac{\partial w}{\partial t}-[\rho]\frac{\partial(wu_\delta)}{\partial t}\right)=[u]\left([\rho v]\frac{\partial w}{\partial t}-[\rho]\frac{\partial(wv_\delta)}{\partial t}\right), \tag{50a}$$

which is rewritten as

$$\frac{\partial}{\partial t}\{([v][\rho u]-[u][\rho v]-[\rho]([v]u_\delta-[u]v_\delta))w\}=0. \tag{50b}$$

It follows that

$$([v][\rho u]-[u][\rho v]-[\rho]([v]u_\delta-[u]v_\delta))w=0.$$

Since $w>0$ when $t>0$ and $[\rho]\neq 0$, we obtain

$$v_2u_1-v_1u_2-[v]u_\delta+[u]v_\delta=0. \tag{51}$$

That is,

$$u_\delta=\frac{[u]}{[v]}u_\delta+\frac{u_1v_2-u_2v_1}{[v]}. \tag{52}$$

Eq. (51) also shows that

$$v_2u_1-v_1u_2-[v]\frac{\partial x}{\partial t}+[u]\frac{\partial y}{\partial t}=0.$$

Integrating it, we get

$$-[v]x+[u]y=-(u_1v_2-u_2v_1)t-[v]s+[u]f(s), \tag{53}$$

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which is differentiated with respect to s, and is

$$-[v]\frac{\partial x}{\partial x}+[u]\frac{\partial y}{\partial s}=-[v]+[u]f'(s).$$

Hence we get

$$\frac{\partial x}{\partial s}=\frac{[u]}{[v]}\frac{\partial y}{\partial s}-\left(\frac{[u]}{[v]}f'(s)-1\right). \tag{54}$$

We calculate

$$u_\delta\frac{\partial y}{\partial s}-v_\delta\frac{\partial x}{\partial s}=\left(\frac{[u]}{[v]}v_\delta+\frac{u_1v_2-u_2v_1}{[v]}\right)\frac{\partial y}{\partial s}-v_\delta\left(\frac{[u]}{[v]}\frac{\partial y}{\partial s}-\left(\frac{[u]}{[v]}f'(s)-1\right)\right)$$

$$=\frac{u_1v_2-u_2v_1}{[v]}\frac{\partial y}{\partial s}+v_\delta\left(\frac{[u]}{[v]}f'(s)-1\right). \tag{55}$$

Therefore, we have

$$\begin{aligned}\frac{\partial w}{\partial t}&=\left(\frac{u_1v_2-u_2v_1}{[v]}\frac{\partial y}{\partial s}+v_\delta\left(\frac{[u]}{[v]}f'(s)-1\right)\right)[\rho]-[\rho u]\frac{\partial y}{\partial s}\\&\quad+[\rho v]\left(\frac{[u]}{[v]}\frac{\partial y}{\partial s}-\left(\frac{[u]}{[v]}f'(s)-1\right)\right)\\&=\left(\frac{u_1v_2-u_2v_1}{[v]}-[\rho u]+\frac{[\rho v][u]}{[v]}\right)\frac{\partial y}{\partial s}+\left(\frac{[u]}{[v]}f'(s)-1\right)([\rho]v_\delta-[\rho v])\\&=\left(\frac{[u]}{[v]}f'(s)-1\right)([\rho]v_\delta-[\rho v]).\end{aligned} \tag{56}$$

Always regarding v_δ as $\frac{\partial y}{\partial t}$, we get

$$\frac{\partial}{\partial t}\left(w-\left(\frac{[u]}{[v]}f'(s)-1\right)[\rho]y\right)=-\left(\frac{[u]}{[v]}f'(s)-1\right)[\rho v].$$

It follows that

$$\begin{aligned}w-\left(\frac{[u]}{[v]}f'(s)-1\right)[\rho]y&=-\left(\frac{[u]}{[v]}f'(s)-1\right)[\rho v]t-\left(\frac{[u]}{[v]}f'(s)-1\right)[\rho]f(s)\\&=-\left(\frac{[u]}{[v]}f'(s)-1\right)([\rho v]t+[\rho]f(s)).\end{aligned} \tag{57}$$

Similarly, we have

$$\frac{\partial(wv_\delta)}{\partial t}=\left(\frac{[u]}{[v]}f'(s)-1\right)([\rho v]v_\delta-[\rho v]^2),$$

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So

$$wv_\delta - \left(\frac{[u]}{[v]}f'(s) - 1\right)[\rho v]y = -\left(\frac{[u]}{[v]}f'(s) - 1\right)([\rho v^2]t + [\rho v]f'(s)). \tag{58}$$

From (57) and (58), we obtain

$$\left(\frac{[u]}{[v]}f'(s) - 1\right)([\rho]v_\delta y - [\rho v]y) = \left(\frac{[u]}{[v]}f'(s) - 1\right)\{([\rho v]t + [\rho]f(s))v_\delta - ([\rho v^2]t + [\rho v]f(s))\}.$$

That is,

$$[\rho]v_\delta y - [\rho v]y = ([\rho v]t + [\rho]f(s))v_\delta - ([\rho v^2]t + [\rho v]f(s)), \tag{59}$$

since $\frac{[u]}{[v]}f'(s) - 1 \neq 0$. Note that (59) is equivalent to

$$\frac{\partial}{\partial t}\left(\frac{[\rho]^2}{2}y^2 - ([\rho v]t + [\rho]f(s))y\right) = -([\rho v^2]t + [\rho v]f(s)).$$

Integrating it, we have

$$\frac{[\rho]^2}{2}y^2 - ([\rho v]t + [\rho]f(s))y + \frac{[\rho]}{2}f^2(s) + \frac{[\rho v^2]}{2}t^2 + [\rho v]f(s)t = 0. \tag{60}$$

Solving (60), we get

$$y = \frac{1}{[\rho]}([\rho v] \pm \sqrt{\rho_1\rho_1}[v])t + f(s). \tag{61}$$

And from (53) we have

$$x = \frac{1}{[\rho]}([\rho u] \pm \sqrt{\rho_1\rho_1}[u])t + s. \tag{62}$$

Thus

$$(u_\delta, v_\delta) = \left(\frac{\sqrt{\rho_1}u_1 \pm \sqrt{\rho_2}u_2}{\sqrt{\rho_1} \pm \sqrt{\rho_2}}, \frac{\sqrt{\rho_1}v_1 \pm \sqrt{\rho_2}v_2}{\sqrt{\rho_1} \pm \sqrt{\rho_2}}\right).$$

By the entropy condition (17), we have

$$u_2 f'(x) - v_2 < u_\delta f'(x) - v_\delta < u_1 f'(x) - v_1. \tag{63}$$

It is easy to check that

$$(u_\delta, v_\delta) = \left(\frac{\sqrt{\rho_1}u_1 - \sqrt{\rho_2}u_2}{\sqrt{\rho_1} - \sqrt{\rho_2}}, \frac{\sqrt{\rho_1}v_1 - \sqrt{\rho_2}v_2}{\sqrt{\rho_1} - \sqrt{\rho_2}}\right) \tag{64}$$

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does not satisfy (63). Thus we choose

$$(u_\delta, v_\delta) = \left(\frac{\sqrt{\rho_1}u_1 + \sqrt{\rho_2}u_2}{\sqrt{\rho_1} + \sqrt{\rho_2}}, \frac{\sqrt{\rho_1}v_1 + \sqrt{\rho_2}v_2}{\sqrt{\rho_1} + \sqrt{\rho_2}}\right) \tag{65}$$

as our solution. So, we have

$$(x, y) = \left(\frac{\sqrt{\rho_1}u_1 + \sqrt{\rho_2}u_2}{\sqrt{\rho_1} + \sqrt{\rho_2}}t + s, \frac{\sqrt{\rho_1}v_1 + \sqrt{\rho_2}v_2}{\sqrt{\rho_1} + \sqrt{\rho_2}}t + f(s)\right). \tag{66}$$

Finally, we obtain

$$\begin{aligned} w(t, s) &= \left(\frac{[u]}{[v]}f'(s) - 1\right)([\rho]v_\delta - [\rho v])t \\ &= \sqrt{\rho_1\rho_2}(-[u]f'(s) + [v]). \end{aligned} \tag{67}$$

Case 6. $[v] - [u]f'(x) < 0$. For this case, the structure of solution is simple and the solution can be written as

$$(\rho, u, v)(t, x, y) = \begin{cases} (\rho_1, u_1, v_1), & x \geqslant x_1, \\ \text{Vac}, & x_2 < x < x_1, \\ (\rho_2, u_2, v_2), & x < x_2, \end{cases} \tag{68}$$

for any fixed $t = T > 0$ and $y = Y$, where x_1 and x_2 are the intersection points of the straight line $t = T$, $y = Y$ and the surfaces $y = f(x - tu_1) + tv_1$ and $y = f(x - tu_2) + tv_2$ respectively.

Remark. For the general case, the structure of solution is just the combination of solution of Cases 5 and 6.

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